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The perfect action for non-degenerate staggered fermions

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Abstract

The perfect action of free staggered fermions is calculated by blocking from the continuum for degenerate and non-degenerate flavor masses. The symmetry structure, connecting flavor transformations and translations, is explained directly from the blocking scheme. It is convenient to use a modified Fourier transformation, respecting this connection, to treat the spin-flavor structure of the blockspins. The perfect action remains local in the non-degenerate case; it is explicitly calculated in two dimensions. I finally comment on the relation of the blocking scheme to the transition from Dirac-Kähler fermions to staggered fermions.

1 Introduction

The lattice formulation of fermions is still subject to an intense debate under two major questions: The first one concerns the improvement of fermion actions and operators to reduce lattice artefacts in a systematic way [1]. Secondly the formulation of gauged chiral fermions exhibits severe problems [2]. They also show up in vector-like theories – writing down a lattice fermion action, one must choose between explicit breaking of chiral symmetry, unwanted doublers, or non-local actions. This is summarized in the no-go theorem of Nielsen and Ninomiya [3].

In contrast to lattice gauge fields the formulation of lattice fermions by a systematic geometric concept is only given for staggered fermions [4]. A root of the continuum Laplacian is defined acting on the space of inhomogeneous differential forms [5]. This Dirac-Kähler (DK) operator is equivalent to the ordinary (free) Dirac operator acting on $2^{d/2}$ flavors of spinor fields. The lattice formulation is obtained by the transition from forms in the continuum to cochains on the lattice, leading to the staggered fermion action [6]. It does not produce any doublers, while preserving part of the chiral symmetry. In view of the above no-go theorem this is possible because of the flavor degeneration of the continuum theory taken as starting point.

A different way to a better understanding of lattice fermion actions can be obtained by the renormalization group (RG) [7]. This approach has been worked out to a large extent in [8, 9]. Starting from a particular original (nearest neighbor) lattice action, an infinite number of RG transformations (RGTs) leads to a perfect action. Provided they preserve the symmetry structure of the original action, this structure holds in the perfect action too. Moreover, if the RGTs show an additional chiral symmetry the question of chiral symmetry restoration in the continuum limit can be analyzed [8, 9]. The crucial point is whether the resulting perfect action remains local (decreases exponentially). If this is the case, ultra-local actions (with a finite range of non-vanishing couplings) might be constructed by truncation of the perfect action, with the symmetry structure determined by the corresponding RGT.

The RG approach becomes more transparent if the perfect action is directly constructed by blocking from the continuum [10]. One might directly design a lattice action with a particular

symmetry structure by an appropriate blocking scheme. The Nielsen-Ninomiya no-go theorem decides whether the resulting perfect action can be expected to be local. In [9] this was studied for the blocking scheme corresponding to Wilson fermions. For staggered fermions the perfect action is calculated in [8] as the fixed point of a lattice blockspin transformation, originally proposed in [11].

I will calculate the perfect staggered fermion action directly by blocking the continuum fermion fields, as proposed already in [12]. After the introduction of the blocking scheme in Section 2 and its symmetries in Section 3 this is performed in Section 4 for a degenerate mass term, corresponding to standard staggered fermions. The main technical ingredient is a modified Fourier transformation for the blockspin variables, consistent with the discrete remnants of flavor symmetry. For two dimensions a perfect action for $2^{d/2}$ fermions was constructed by blocking from the continuum in a different way, which does not directly correspond to staggered fermions [13].

A generalization of staggered fermions to non-flavor-degenerate mass terms is desirable for a physical interpretation of the $2^{d/2}$ flavors. This was discussed already in [14, 15]. In Section 5 I calculate the perfect action using the blocking scheme of Section 2, yet with a continuum action with non-degenerate flavor–dependence. This perfect action can be shown to be local, and thus may be truncated to be used in simulations of flavors with different masses. In Section 6 I discuss this action in more detail for two dimensions. Here I show even-odd decoupling for $m^\dagger m$, m is the fermion matrix. Therefore numerical calculations using the pseudofermion method [17] are possible without an additional flavor doubling, as for standard staggered fermions. A detailed study of the four-dimensional case shall be subject of a forthcoming paper.

The blocking scheme closely resembles the cochain construction of staggered fermions from DK fermions, which can itself be looked at as a blocking procedure with partial decimation. In fact, the cochain construction served as a guideline for the staggered fermion blockspin transformation originally proposed in [11]. In Section 7 I discuss this similarity. However, an attempt to construct an alternative perfect action using this cochain construction fails.

2 The blocking scheme

The starting point of the perfect action with staggered fermion symmetry are $N_f = 2^{d/2}$ flavors of continuum fermion fields $\psi_a^b(x)$, with a and b the spinor and flavor index, respectively. The action reads

$$S[\bar{\psi},\psi] \;=\; \int dx\; \bar{\psi}_a^b(x)\,(\gamma^\mu_{aa'}\partial_\mu \;+\; \delta_{aa'}m)\,\delta_{bb'}\;\psi_{a'}^{b'}(x)\;. \tag{1}$$

Summation over double indices is understood, x is the d-dimensional continuous space coordinate. One conveniently defines the component functions of inhomogeneous differential forms

$$\Phi \;=\; \sum_H \varphi(x,H)dx^H\;,\quad dx^H \;=\; dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_h}\;,\quad \mu_1<\ldots<\mu_h\;. \tag{2}$$

This defines H as a set of h indices (multi-index) $H=\{\mu_1,\ldots,\mu_h\}$. As described in Section 7, this is the starting point for DK fermions [5], and the whole procedure bears a strong resemblance to the mapping of forms onto lattice cochains leading from the continuum DK equation to staggered fermions on the lattice [4]. I disregard this point up to Section 7 and treat the following equations as a mere prescription for a particular kind of blockspin definition (looking unnecessarily complicated). The $2^{d/2}$ Dirac spinors are unitarily transformed to the component

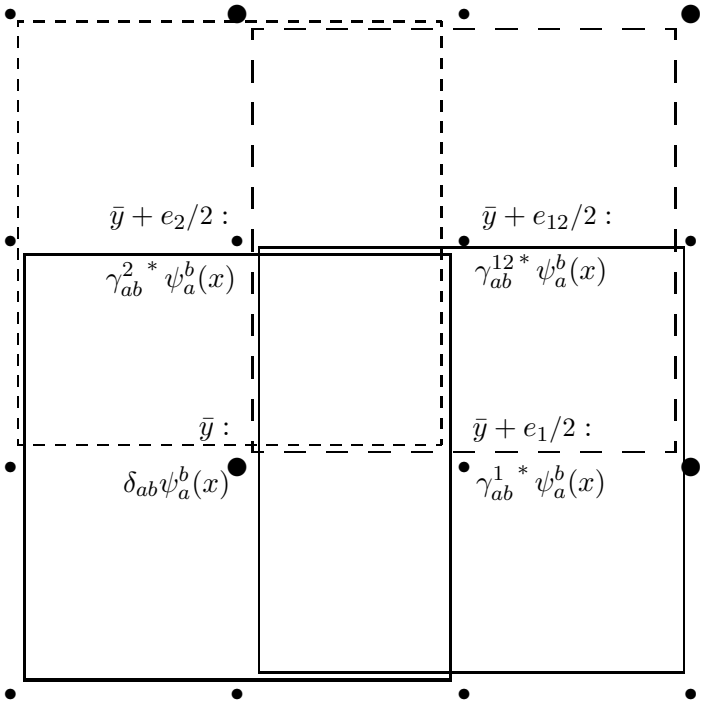

Figure 1: Blocking scheme for $d = 2$. The coarse lattice points $\bar{y}$ are marked by larger circles. The blockspin at point $\bar{y} + e_H/2$ is the average of the specified projection of $\psi^b_a(x)$ in spin–flavor space, in the block surrounding that point.

functions $\varphi(x, H)$, with $H = \{\mu_1, \ldots, \mu_h\}$. Transformation and backtransformation read

$$\varphi(x,H) \;=\; \frac{1}{\sqrt{N_f}} \sum_{ab} \gamma^{H\,*}_{ab}\ \psi^b_a(x)\ , \quad \gamma^H \;=\; \gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_h} \tag{3}$$

$$\psi^b_a(x) \;=\; \frac{1}{\sqrt{N_f}} \sum_{H} \gamma^H_{ab}\, \varphi(x,H)\ . \tag{4}$$

In order to verify the backtransformation use the orthogonality and completeness relations

$$\sum_H \gamma^H_{ab}\ \gamma^{H\ *}_{a'b'} \;=\; N_f\ \delta_{aa'}\,\delta_{bb'}\ , \tag{5}$$

$$\sum_{ab} \gamma^H_{ab}\ \gamma^{K\,*}_{ab} \;=\; N_f\ \delta_{HK}\ . \tag{6}$$

As further ingredients to the blockspin definition one needs a coarse lattice $\bar{\Gamma} = \{\bar{y}\,|\,\bar{y}_\mu = a\bar{n}_\mu\}$, and a fine lattice $\Gamma = \{y\,|\,y_\mu = (a/2)n_\mu\}$, with $\bar{n}_\mu, n_\mu \in \mathbb{Z}$. The fine lattice points y are uniquely decomposed as (e_μ is the vector of length a in μ–direction)

$$y \;=\; \bar{y} \,+\, e_H/2\ , \quad e_H \;=\; \sum_{\mu\in H} e_\mu\ . \tag{7}$$

This defines the multi-index $H(y)$ as position of the fine lattice point y with respect to the coarse lattice $\bar{\Gamma}$. The blockspin variables $\phi(y)$ are now defined as averages of the component

functions $\varphi(x, H(y))$ over the lattice hypercubes $[y] = \{x \mid -a/2 \leq x_\mu - y_\mu \leq a/2\}$, as proposed already in [12],

$$\phi(y) \;=\; \frac{1}{a^d}\int_{[y]} dx\; \varphi(x, H(y)) \;=\; \frac{1}{a^d\sqrt{N_f}}\sum_{ab} \gamma_{ab}^{H(y)\,*} \int_{[y]} dx\; \psi_a^b(x)\;, \tag{8}$$

see Figure 1. This means for fixed H the component functions are blocked onto the coarse lattice, the block centers, however, are staggered depending on the multi-index H, leading to one-component blockspins on the fine lattice. Staggering the block centers is essential. If the coarse grid points were used as blocking centers for blockspins $\tilde{\phi}(\bar{y}, H)$ this procedure would commute with the backtransformation Eq. (4) to the spinor basis, leading to a separate blocking of the N_f flavors. As pointed out in [8] such a blocking scheme leads to a local perfect action, only if an additional chiral symmetry breaking term is included, whereas with Eq. (8) part of the chiral symmetry survives, and the perfect action will turn out to be local.

3 Blockspin symmetry

The blocking scheme determines the symmetry of the perfect action simply by induction of continuum symmetries consistent with the definition of the blockspin variables. In particular, from Eq. (8) one is lead to the symmetries of staggered fermions [15, 18]. It is evident from the block sizes determined by the coarse lattice that the blockspin action will be invariant under translations on the coarse lattice. The restriction of flavor and chiral symmetry is somewhat more involved. In the continuum these transformations are generated by

$$\text{flavor transformation:} \qquad \psi'^b_a(x) \;=\; \psi^{b'}_a(x)\; \gamma^{K\;\dagger}_{b'b} \tag{9}$$

$$\text{chiral transformation:} \qquad \psi'^b_a(x) \;=\; \psi^b_{a'}(x)\gamma^5_{aa'}\;. \tag{10}$$

In general flavor transformations are not consistent with the blocking scheme Eq. (8). Formally they would induce a transformation of the blockspins

$$\phi'(y) \;=\; \rho(H,K)\;\frac{1}{N_f\,a^d}\sum_{ab} \gamma_{ab}^{H(y)\Delta K\,*} \int_{[y]} dx\; \psi_a^b(x)\;. \tag{11}$$

The sign functions $\rho(H, K)$ are defined such that

$$\gamma^H\,\gamma^K \;=\; \rho(H,K)\;\gamma^{H\Delta K}\;, \tag{12}$$

the symmetric difference $H\Delta K = (H\cup K)\backslash(H\cap K)$ fulfills $H(y\pm y') = H(y)\Delta H(y')$. One easily proves

$$\rho(H\Delta H', K) \;=\; \rho(H,K)\;\rho(H',K)\;,\quad \rho(H, K\Delta K') \;=\; \rho(H,K)\;\rho(H,K')\;. \tag{13}$$

However, $\phi'(y)$ is not a proper blockspin variable, because the multi-index $H(y)\Delta K$ of the γ–matrix in Eq. (11) is not equal to the multi-index $H(y)$ of the block center $[y]$. For the discrete transformations in Eq. (9) (now considered as finite unitary transformations) this can be cured by a combination with an appropriate fine lattice shift $x \to x - e_K/2$. The corresponding transformation of the blockspins becomes

$$d^K\,\phi(y) \;=\; \rho(H(y), K)\;\phi(y + e_K/2)\;. \tag{14}$$

The discrete chiral transformation combined with a shift $x \to x - e_5/2$ ($5 \equiv \{1234\}$ for $d = 4$) reads

$$C\,\phi(y) \;=\; \rho(5, H(y))\;\phi(y + e_5/2)\;. \tag{15}$$

Now consider the combination of chiral transformation Eq. (10) and discrete flavor transformation given by ${\gamma^5_{bb'}}^\dagger$ in Eq. (9). In the $\varphi(x, H)$–basis it simply acts as a multiplication by $(-1)^h$. Therefore it generates transformations, also defined for the blockspins

$$(c(\alpha)\,\phi)\,(\bar{y} + e_H/2) \;=\; e^{i\alpha(-1)^h}\;\phi(\bar{y} + e_H/2)\;, \tag{16}$$

corresponding to the continuous even–odd symmetry of staggered fermions.

Since fine lattice shifts must be combined with flavor transformations to yield a symmetry of the blockspin theory, ordinary Fourier transformation with momenta chosen in the fine lattice Brillouin zone is not convenient for a diagonalization of propagator and action. Instead, the correct basis transformation to this end is given by the lattice fields realizing the irreducible representations of coarse lattice translations and discrete flavor transformations [19]. They do not commute, thus the irreducible representations (with non-trivial representation of the sign factor) are multi-dimensional. They are labeled by a momentum p in the coarse lattice Brillouin zone $\mathcal{B}$: $-\pi/a \le p_\mu < \pi/a$. I call the corresponding basis transformation symmetry consistent Fourier transformation (scFT)

$$\phi^b_a(p) \;=\; \sum_y e^{ipy}\,{\gamma^{H(y)}_{ab}}\,\phi(y)\;. \tag{17}$$

Using Eqs. (5, 6) the backtransformation is

$$\phi(y) \;=\; \frac{1}{N_f}\int_{\mathcal{B}} \frac{dp}{(2\pi/a)^d}\;\sum_{ab} e^{-ipy}\;{\gamma^{H(y)}_{ab}}^*\,\phi^b_a(p)\;. \tag{18}$$

The discrete (modified) flavor transformations d^K take the form

$$(d^K\phi)^b_a(p) \;=\; e^{ip\,e_K/2}\;\phi^{b'}_a(p)\;\;{\gamma^K_{b'b}}^\dagger\;. \tag{19}$$

Their meaning, a combined flavor transformation and translation, can be read off directly.

For the lattice restriction of rotations and reflections the same considerations as for flavor transformations apply. In Eq. (8) the transformation of the blocking cell $[y]$ must be accompanied by the corresponding transformation of ${\gamma^{H(y)}_{ab}}^*$ by a suitable transformation in spin and flavor space. This is given by a combination of spinorial rotations and reflections with flavor transformations, called geometric transformations. In the $\phi^b_a(p)$ basis they are defined by

$$\text{geometric rotation by } 90^\circ\text{:} \qquad R^{\mu\nu}_G\,\phi^b_a(p) \;=\; \frac{1}{2}(1 + \gamma^{\mu\nu})_{aa'}\;\phi^{b'}_{a'}(R^{-1}_{\mu\nu}p)\;(1 - \gamma^{\mu\nu})_{b'b}\;, \tag{20}$$

$$\text{geometric reflection:} \qquad \Pi^\mu_G\,\phi^b_a(p) \;=\; \gamma^{\bar{\mu}}_{aa'}\;\phi^{b'}_{a'}(\Pi^\mu p)\;\;{\gamma^{\bar{\mu}}_{b'b}}^\dagger\;, \tag{21}$$

with $\bar{\mu} = \mu\Delta 1234$, $(R^{-1}_{\mu\nu}p)_\mu = p_\nu$, $(R^{-1}_{\mu\nu}p)_\nu = -p_\mu$, and $(\Pi^\mu p)_\nu = (1 - 2\delta_{\mu\nu})p_\nu$. An analogue to spinorial rotations by 180° and reflections can be defined by combinations with the discrete modified flavor transformations of Eq. (14)

$$\text{spinorial rotation by } 180^\circ\text{:} \qquad \bar{R}^{\mu\nu}_S\,\phi^b_a(p) \;=\; \gamma^{\mu\nu}_{aa'}\;\phi^b_{a'}(R^{-2}_{\mu\nu}p)\;e^{-ip\,e_K/2}\;, \tag{22}$$

$$\text{spinorial reflection:} \qquad \Pi^\mu_S\,\phi^b_a(p) \;=\; \gamma^{\bar{\mu}}_{aa'}\;\phi^b_{a'}(\Pi^\mu p)\;e^{-ip\,e_{\bar{\mu}}/2}\;. \tag{23}$$

The elimination of the flavor transformation is paid by the combination with a translation, in order to fit the blocking scheme of Eq. (8).

4 The perfect action

Everything to be done in a free theory is the calculation of the blockspin propagator $u(y,y') \equiv \langle \phi(y)\bar{\phi}(y')\rangle$ using the continuum action $S[\bar{\psi},\psi]$, and the definition of the blockspins $\phi(y)$ by the continuum fields $\psi_a^b(x)$. The perfect blockspin action is then given by the inverse propagator $m = u^{-1}$

$$S_{\mathit{eff}}[\bar{\phi},\phi] \;=\; \sum_{y,y'} \bar{\phi}(y)\, m(y,y')\, \phi(y') \;, \tag{24}$$

i.e. it reproduces $u(y,y')$ in the lattice path integral. Let me first evaluate

$$\langle\; \frac{1}{a^d}\int_{[y]} dx\; \psi_a^b(x)\; \frac{1}{a^d}\int_{[y']} dx'\; \bar{\psi}_{a'}^{b'}(x')\;\rangle \;=\; \delta_{bb'}\; \tilde{U}_{aa'}(y-y') \;. \tag{25}$$

It results in

$$\tilde{U}_{aa'}(y) \;=\; \frac{1}{(2\pi)^d}\int dp\; e^{-ipy}\; (i\gamma^\mu p_\mu \;+\; m)_{aa'} R(m,p) \;, \tag{26}$$

$$R(m,p) \;=\; \frac{1}{p^2+m^2}\prod_\mu \frac{\sin^2(ap_\mu/2)}{(ap_\mu/2)^2} \;. \tag{27}$$

Comparing with Eq. (8) the blockspin propagator becomes

$$u(y,y') \;=\; \rho(y-y',y')\; U(y-y') \;, \tag{28}$$

$$U(y) \;=\; \frac{1}{N_f}\sum_{aa'} \gamma_{aa'}^{H(y)\,*}\; \tilde{U}_{aa'}(y) \;. \tag{29}$$

The sign function in Eq. (28) $\rho(y-y',y') = \rho(H(y-y'),H(y'))$ is defined in Eq. (12). Due to its dependence on y' as second argument it is not invariant under fine lattice translations but under the discrete flavor transformations d^K of Eq. (14). In order to diagonalize propagator and action I therefore use the symmetry consistent Fourier transformation (scFT) in Eqs. (17, 18).

In order to evaluate the perfect action S_{eff}, it is convenient not to invert the propagator directly. Instead, I calculate $U(y)$ by inversion of (the formal expression of) the action, then I compare with Eqs. (29, 26). Due to its symmetry properties $m(y,y')$ can be written as

$$m(y,y') \;=\; \rho(y-y',y')\; M(y-y') \;. \tag{30}$$

Inserting the inverse scFT Eq. (18) I obtain

$$\begin{aligned} S_{\mathit{eff}} \;&=\; \frac{1}{N_f^2}\int_{\mathcal{B}} \frac{dp}{(2\pi/a)^d}\int_{\mathcal{B}} \frac{dp'}{(2\pi/a)^d}\; \sum_{ab}\sum_{a'b'}\; \bar{\phi}_a^b(p)\, \phi_{a'}^{b'}(p') \\ &\times\; \sum_z e^{ipz}\, \gamma_{ac}^{H(z)}\, M(z)\, X_{ca'}^{bb'}(p-p') \;. \end{aligned} \tag{31}$$

With

$$X_{ca'}^{bb'}(p-p') \;\equiv\; \sum_{y'} e^{i(p-p')y'}\, \gamma_{cb}^{H(y')}\, \gamma_{a'b'}^{H(y')\,*} \;=\; N_f\; \delta_{ca'}\; \delta_{bb'}\; (2\pi/a)^d \delta(p-p') \;, \tag{32}$$

following from Eq. (5) and the restriction of p, p' to the coarse Brillouin zone $\mathcal{B}$, I arrive at

$$S_{\mathit{eff}} \;=\; \frac{1}{N_f}\int_{\mathcal{B}}\frac{dp}{(2\pi/a)^d}\;\sum_{ab}\sum_{a'b'}\;\bar{\phi}^b_a(p)\,\phi^{b'}_{a'}(p)\;\delta_{bb'}\,M_{aa'}(p)\;, \tag{33}$$

$$M_{aa'}(p) \;=\; \sum_y e^{ipy}\,\gamma^{H(y)}_{aa'}\,M(y)\;. \tag{34}$$

The propagator of the symmetry consistent Fourier transformed fields is then

$$\langle\phi^b_a(p)\,\bar{\phi}^{b'}_{a'}(p')\rangle \;=\; N_f\frac{2\pi^d}{a^d}\;\delta(p-p')\;\delta_{bb'}\;M^{-1}_{aa'}(p)\;, \tag{35}$$

and with the inverse scFT in Eq. (18)

$$U(y) \;=\; \langle\phi(y)\,\bar{\phi}(0)\rangle \;=\; \frac{1}{N_f}\int_{\mathcal{B}}\frac{dp}{(2\pi/a)^d}\;\sum_{aa'}\,e^{-ipy}\;\gamma^{H(y)\,*}_{aa'}\;M^{-1}_{aa'}(p)\;. \tag{36}$$

One has now the desired equation to compare with Eqs. (29, 26). It follows

$$M^{-1}_{aa'}(p) \;=\; \frac{1}{a^d}\left(i\gamma^\mu_{aa'}Q_\mu(p)\;+\;\delta_{aa'}\,Q_0(p)\right)\;, \tag{37}$$

$$Q_\mu(p) \;=\; \sum_{k\in\mathbb{Z}^d}(-1)^{k_\mu}\,(p_\mu+\frac{2\pi}{a}k_\mu)\,R(m,p+\frac{2\pi}{a}k)\;, \tag{38}$$

$$Q_0(p) \;=\; m\sum_{k\in\mathbb{Z}^d}R(m,p+\frac{2\pi}{a}k)\;. \tag{39}$$

Inversion and an inverse scFT of $M_{aa'}(p)$, see Eq. (34), lead to

$$M(y) \;=\; \frac{a^d}{N_f}\int_{\mathcal{B}}\frac{dp}{(2\pi/a)^d}\;\sum_{aa'}e^{-ipy}\,\gamma^{H(y)\,*}_{aa'}\,\frac{-i\gamma^\mu_{aa'}Q_\mu(p)\;+\;\delta_{aa'}\,Q_0(p)}{Q^\mu(p)Q_\mu(p)+Q_0(p)^2}\;. \tag{40}$$

Using Eq. (6) a non-vanishing $M(y)$ appears only for certain positions of $y=\bar{y}+e_H/2$ with respect to the coarse lattice points $\bar{y}$. The final result for the perfect action is therefore

$$S_{\mathit{eff}}[\bar{\phi},\phi] \;=\; \sum_{y,y'}\;\bar{\phi}(y)\;\rho(y-y',y')\;M(y-y')\;\phi(y')\;, \tag{41}$$

$$M(\bar{y}+e_\mu/2) \;=\; a^d\int_{\mathcal{B}}\frac{dp}{(2\pi/a)^d}\,e^{-ip\bar{y}}\;\;\frac{-iQ_\mu(p)\,e^{-ia\,p_\mu/2}}{Q^\mu(p)Q_\mu(p)+Q_0(p)^2}\;, \tag{42}$$

$$M(\bar{y}) \;=\; a^d\int_{\mathcal{B}}\frac{dp}{(2\pi/a)^d}\,e^{-ip\bar{y}}\;\;\frac{Q_0(p)}{Q^\mu(p)Q_\mu(p)+Q_0(p)^2}\;, \tag{43}$$

$$M(\bar{y}+e_H/2) \;=\; 0\;\;\text{for}\;H\neq\mu,\emptyset\;. \tag{44}$$

It has the structure of the staggered fermion action [6]. The latter arises in the above notation by restriction of $M(y-y')$ to the nearest neighbor couplings $M_{KS}(y-y')$

$$M_{KS}(\pm e_\mu/2) \;=\; \mp\,a^{d-1}\;,\qquad M_{KS}(0) \;=\; m\,a^d\;. \tag{45}$$

The result of Eqs. (41–44) was achieved in [8][1] as fixed point action of a RGT from a fine to a coarse lattice proposed in [11], see also the third paper of [1]. This RGT commutes with the blocking scheme from the continuum to fine and coarse lattice, respectively. The locality of the perfect action can be read off from the analyticity of the Fourier transformed p-dependent fractions in Eqs. (42, 43) and their periodicity with respect to the coarse Brillouin zone $\mathcal{B}$.

[1] The authors considered a generalization by gaussian smearing of the blocking δ-function.

5 Non-degenerate flavors

Let me apply the blocking scheme of Section 2 to the case of non-degenerate flavors. I assume a continuum propagator with a flavor-dependent mass

$$\langle \psi_a^b(x)\, \bar{\psi}_{a'}^{b'}(0) \rangle \;=\; \delta_{bb'}\; \frac{1}{(2\pi)^d} \int dp\; e^{-ipx}\, \frac{(i\gamma^\mu p_\mu \,+\, m_b)_{aa'}}{p^2 \,+\, m_b^2}\;. \tag{46}$$

The modification of staggered fermions to the latter case is discussed in [14, 15]. The perfect action derived here might be used to explain the necessary structure of such a modification. I proceed in generalization of Section 4 with the definition

$$\langle\; \frac{1}{a^d} \int_{[y]} dx\; \psi_a^b(x)\; \frac{1}{a^d} \int_{[y']} dx'\; \bar{\psi}_{a'}^{b'}(x')\; \rangle \;=\; \sum_K \gamma_{bb'}^K\; \tilde{U}_{aa'}^K(y-y')\;, \tag{47}$$

$$\tilde{U}_{aa'}^K(y) \;=\; \frac{1}{(2\pi)^d} \int dp\; e^{-ipy}\; \frac{1}{N_f} \sum_b \gamma_{bb}^{K\,*}\, (i\gamma^\mu p_\mu \,+\, m_b)_{aa'}\, R(m_b,p)\;. \tag{48}$$

Since the propagator is diagonal in flavor space $\tilde{U}_{aa'}^K(y)$ is non-zero iff γ^K is diagonal, denoted by $K \in \mathcal{D}$. In the following this restriction is understood for the sums over K. Corresponding to Eqs. (28, 29) I find

$$u(y,y') \;=\; \sum_K \rho^K(y')\rho(y-y',y')\; U^K(y-y')\;, \tag{49}$$

$$U^K(y) \;=\; \frac{1}{N_f} \sum_{aa'} \gamma_{aa'}^{H(y)\,*}\; \tilde{U}_{ac}^K(y)\; \gamma_{ca'}^{K\;T}\;, \tag{50}$$

$$\rho^K(y') \;=\; \rho(K,H(y'))\, \rho(H(y'),K)\;. \tag{51}$$

Inserting Eq. (48) $U^K(y)$ is cast in the form of an inverse scFT

$$U^K(y) \;=\; \frac{1}{N_f} \int_{\mathcal{B}} \frac{dp}{(2\pi/a)^d} \sum_{aa'} \gamma_{aa'}^{H(y)\,*}\; e^{-ipy}\; U_{aa'}^K(p)\;, \tag{52}$$

$$U_{aa'}^K(p) \;=\; \frac{1}{a^d} \left(\sum_\mu i(\gamma^\mu\, \gamma^{K\,T})_{aa'}\, Q_\mu^K(p) \;+\; \gamma_{aa'}^{K\;T}\, Q_0^K(p) \right)\;, \tag{53}$$

with

$$Q_\mu^K(p) \;=\; \sum_{k\in\mathbb{Z}^d} \prod_{\nu\in K\Delta\mu} (-1)^{k_\nu}\, \frac{1}{N_f} \sum_b \gamma_{bb}^{K\,*}\; (p_\mu + \frac{2\pi}{a} k_\mu)\, R(m_b, p+\frac{2\pi}{a}k)\;, \tag{54}$$

$$Q_0^K(p) \;=\; \sum_{k\in\mathbb{Z}^d} \prod_{\nu\in K} (-1)^{k_\nu} \frac{1}{N_f} \sum_b \gamma_{bb}^{K\,*}\; m_b\, R(m_b, p+\frac{2\pi}{a}k)\;. \tag{55}$$

The propagator of the symmetry consistent Fourier transformed fields becomes

$$\begin{aligned} \langle \phi_a^b(p)\, \bar{\phi}_{a'}^{b'}(p') \rangle \;&=\; \sum_K \sum_z e^{ipz} \gamma_{ac}^{H(z)} U^K(z)\; \sum_{y'} e^{i(p-p')y'} \gamma_{cb}^{H(y')}\; \gamma_{dd'}^{H(y')\,*}\;\; \gamma_{a'd}^{K\;\dagger}\, \gamma_{d'b'}^K \\ &=\; N_f (2\pi/a)^d\, \delta(p-p')\; \sum_K \gamma_{bb'}^K\; U_{ac}^K(p)\; \gamma_{ca'}^{K\;*}\;, \end{aligned} \tag{56}$$

with use of Eq. (32) and the backtransformation of Eq. (52) for the last line. Finally I obtain

$$\langle \phi^b_a(p)\, \bar\phi^{b'}_{a'}(p')\rangle \;=\; N_f(2\pi/a)^d\, \delta(p-p')\; \sum_K \gamma^K_{bb'}\; V^K_{aa'}(p)\;, \tag{57}$$

$$V^K_{aa'}(p) \;=\; \frac{1}{a^d}\left(\sum_\mu i\gamma^\mu_{aa'}\, Q^K_\mu(p) \;+\; \delta_{aa'} Q^K_0(p)\right)\;. \tag{58}$$

Thus, starting with diagonal γ^K–matrices in flavor space, $K\in\mathcal{D}$, the diagonality is recovered for $\langle \phi^b_a(p)\bar\phi^{b'}_{a'}(p')\rangle$.

The action may be written due to coarse lattice translation symmetry

$$S_{\mathit{eff}}[\bar\phi,\phi] \;=\; \sum_{y,y'} \bar\phi(y)\, m(y,y')\, \phi(y')\;, \tag{59}$$

$$m(y,y') \;=\; \sum_K \rho^K(y')\rho(y-y',y')\; M^K(y-y')\;. \tag{60}$$

The transition to the symmetry consistent Fourier transformed fields yields

$$S_{\mathit{eff}} \;=\; \frac{1}{N_f}\int_{\mathcal{B}} \frac{dp}{(2\pi/a)^d}\; \sum_{ab}\sum_{a'b'}\; \bar\phi^b_a(p)\, \phi^{b'}_{a'}(p)\; \sum_K \gamma^K_{bb'}\; M^K_{ac}(p)\; {\gamma^K_{ca'}}^*\;, \tag{61}$$

$$M^K_{ac}(p) \;=\; \sum_z e^{ipz}\, \gamma^{H(z)}_{ac}\, M^K(z)\;. \tag{62}$$

As propagator and action are diagonal in flavor space one has to sum over diagonal γ^K–matrices only. This corresponds to the remaining discrete flavor symmetry transformations $d^K, K\in\mathcal{D}$, see [14]. It follows that choosing $\mathcal{D}=\{\emptyset, 12, 34, 1234\}$ also the geometric rotations ω^{12},ω^{34} of Eq. (20) remain as a symmetry, whereas in general the rotation and reflection symmetry is kept only in the modified spinorial form of Eqs. (22, 23). Inversion of $\sum_K \gamma^K_{bb'} M^K_{ac}(p)\, {\gamma^K_{ca'}}^*$ and comparison with Eq. (57) lead to

$$\sum_K \rho(K,L)\left(M^K(p)\; {\gamma^K}^T\, V^{K\Delta L}(p)\right)_{aa'} \;=\; \delta_{L,\emptyset}\; \delta_{aa'}\;,\quad K,L\in\mathcal{D}\;. \tag{63}$$

This determines $M^K(p)$ and S_{eff} with use of the backtransformation of Eq. (62)

$$M^K(\bar y + e_H/2) \;=\; \frac{1}{N_f}\int_{\mathcal{B}} \frac{dp}{(2\pi/a)^d}\, e^{-ip\bar y}\; \mathrm{Tr}\left(e^{-ipe_H/2}\; {\gamma^H}^*\, M^K(p)\right)\;. \tag{64}$$

Note that the blockspin propagtor $u(y,y')$ in the non-flavor-degenerate case Eq. (49), and thus the fermion matrix $m(y,y')$, get complex values depending on the multi-indices $H(y), H(y')$.

Let me consider the vicinity of $p=0$. The factor $R(m,p+\frac{2\pi}{a}k)$ becomes (up to corrections of $\mathcal{O}(p^2)$) $(\prod_\mu \delta_{k_\mu,0})/(p^2+m^2)$, and the sums $\sum_{k\in\mathbb{Z}^d}$ in $V^K(p)$ drop out. In this approximation Eq. (5) can be used to simplify Eqs. (57, 58)

$$\langle \phi^b_a(p)\, \bar\phi^{b'}_{a'}(p')\rangle \;=\; N_f(2\pi/a)^d\, \delta(p-p')\; \frac{1}{a^d}\frac{(\sum_\mu i\gamma^\mu\, p_\mu + m_b)_{aa'}}{p^2+m_b^2}\delta_{bb'}\;, \tag{65}$$

and the Fourier representation of S_{eff} in Eq. (61) becomes

$$S_{\mathit{eff}} \;=\; \frac{1}{N_f}\int_{\mathcal{B}} \frac{dp}{(2\pi/a)^d}\; \sum_{ab}\sum_{a'b'}\; \bar\phi^b_a(p)\, \phi^{b'}_{a'}(p)\; a^d\Big[\delta_{bb'}(-i\gamma^\mu p_\mu + m_b)_{aa'} + \mathcal{O}(p^2)\Big]\;. \tag{66}$$

In order to prove S_{eff} to be local, let me define the transformations B^μ in momentum and spinor space ($\hat{\mu}$ is the unit vector in μ–direction of momentum space)

$$B^\mu F_{ab}(p) \;=\; {\gamma^{\bar{\mu}}_{aa'}}^{\dagger}\, F_{a'b'}\,(p+\frac{2\pi}{a}\hat{\mu})\;\gamma^{\bar{\mu}}_{b'b}\;, \tag{67}$$

$\bar{\mu} = 1234\Delta\mu$ $(12\Delta\mu)$ for $d = 4\,(2)$. Since $(B^\mu)^2 V^K(p) = V^K(p)$ thus $(B^\mu)^2 M^K(p) = M^K(p)$, one may decompose $M^K(p)$ with respect to its behavior under these transformations

$$M^K(p) \;=\; \sum_H m_H^K(p)\;, \qquad \text{with}\;\; B^\mu\, m_H^K(p) \;=\; \sigma_H^\mu\, m_H^K(p)\;. \tag{68}$$

Here $\sigma_H^\mu = -1\,(1)$ for $\mu \in H\,(\mu \notin H)$, and the components $m_H^K(p)$ are uniquely determined. With $B^\mu\,\gamma^K = \sigma_K^\mu\,\gamma^K$ and $B^\mu\,V^K(p) = \sigma_K^\mu\,V^K(p)$ it follows from Eq. (63)

$$\sum_K \rho(K,L) \sum_H \sigma_H^\mu\,\sigma_L^\mu \left(m_H^K(p)\;{\gamma^K}^T\,V^{K\Delta L}(p)\right)_{aa'} \;=\; \delta_{L,\emptyset}\;\delta aa'\;. \tag{69}$$

This is solved by $m_H^K(p) = \delta_{H,\emptyset}\,M^K(p)$ giving back Eq. (63). Thus $M^K(p)$ is invariant under B^μ, as is the term $e^{-ipe_H/2}\;{\gamma^H}^*$ in the trace of Eq. (64), and this trace is periodic with respect to the Brillouin zone $\mathcal{B}$. On the other hand $M^K(p)$ is analytic for all $p \in \mathcal{B}$, because $V^K(p)$ has no zeros in $\mathcal{B}$. In conclusion the perfect action remains local in the non-degenerate case[2]. I will analyze the structure of this action in a simple case in the following section.

6 The two-dimensional case

As example I study the case of $d = 2$ with fermion masses m_1, m_2. I choose $\gamma^{12} = diag(i, -i)$, thus $\mathcal{D} = \{\emptyset,\, 12\}$, and I obtain

$$V^\emptyset_{aa'}(p) \;=\; \frac{1}{2a^d}\sum_{k\in\mathbb{Z}^d}\Big[\tilde{R}^+(\tilde{p})\,\delta_{aa'} \;+\; \bar{R}^+(\tilde{p})\sum_\mu (-1)^{k_\mu}\,i\gamma^\mu_{aa'}\,\tilde{p}_\mu\Big]\;, \tag{70}$$

$$V^{12}_{aa'}(p) \;=\; \frac{-i}{2a^d}\sum_{k\in\mathbb{Z}^d}\Big[\tilde{R}^-(\tilde{p})\,\delta_{aa'} \;+\; \bar{R}^-(\tilde{p})\sum_\mu (-1)^{k_\mu}\,i\gamma^\mu_{aa'}\,\tilde{p}_\mu\Big]\,\epsilon(k)\;, \tag{71}$$

with $\tilde{p} = p + \frac{2\pi}{a}k$, $\bar{R}^\pm(\tilde{p}) = R(m_1,\tilde{p}) \pm R(m_2,\tilde{p})$, $\tilde{R}^\pm(\tilde{p}) = m_1 R(m_1,\tilde{p}) \pm m_2 R(m_2,\tilde{p})$, and $\epsilon(k) = (-1)^{k_1+k_2}$. For simplicity of notation I furthermore define

$$V^\emptyset(p) \;=\; A \;=\; \gamma^\mu a_\mu + \mathbb{1}a_0\;,\;\; V^{12}(p) \;=\; B \;=\; \gamma^\mu b_\mu + \mathbb{1}b_0\;, \tag{72}$$

$$(a,b) \;=\; a_1b_1 + a_2b_2 - a_0b_0\;,\;\; (a,\epsilon b) \;=\; a_1b_2 - a_2b_1\;, \tag{73}$$

$$c_\mu \;=\; a_\mu b_0 - a_0 b_\mu\;. \tag{74}$$

With $\bar{A} = \gamma^\mu a_\mu - a_0$ it follows $\bar{A}A = (a,a) \equiv a^2$, and $\bar{A}B \;=\; (a,b)\mathbb{1} + (a,\epsilon b)\gamma^{12} + c_\mu\gamma^\mu$. Now Eq. (63) reads

$$M^\emptyset(p)\,A \;+\; M^{12}(p)\,{\gamma^{12}}^T\,B \;=\; \mathbb{1}\;, \qquad M^\emptyset(p)\,B \;-\; M^{12}(p)\,{\gamma^{12}}^T\,A \;=\; 0\;. \tag{75}$$

[2] After completion of this paper a calculation of the perfect action of degenerate staggered fermions by blocking from the continuum was published in [16]. The authors used a smeared blockspin transformation within a somewhat different calculation scheme. Optimizing the additional smearing parameters the exponential decay constant of the couplings can be increased considerably. This can be performed also in the non-degenerate case. For the same choice of smearing parameters the impact on the locality is roughly the same as in [16].

$m(y,0), m(0,y)$	$y_1 = 0$	$y_1 = 1/2$	$y_1 = 1$	$y_1 = 3/2$
$y_2 = 0$	0.814	∓ 1.915	-0.054	± 0.308
$y_2 = 1/2$	∓ 1.915	$\mp 0.172\, i$	$\pm 0.118 + 0.004\, i$	$\pm 0.048\, i$
$y_2 = 1$	-0.054	$\pm 0.118 - 0.004\, i$	0.002	$\mp 0.005 + 0.001\, i$
$y_2 = 3/2$	± 0.308	$\pm 0.048\, i$	$\mp 0.005 - 0.001\, i$	$\mp 0.007\, i$
$y_2 = 2$	-0.003	$\pm 0.006 + 0.003\, i$	-0.001	$\mp 0.002 - 0.0004\, i$
$y_2 = 5/2$	∓ 0.046	$\mp 0.011\, i$	$\mp 0.003 + 0.0002\, i$	$\pm 0.0002\, i$

Table 1: The first couplings of the fermion matrix $m(y,0), m(0,y)$. If forward and backward coupling differ, the upper sign belongs to $m(y,0)$.

This is solved by

$$\begin{aligned} M^{\emptyset}(p) &= N^{-1}\bar{A}\ , & (76)\\ N &= \bar{A}A + \bar{A}BA^{-1}B \ = \ a^2 + \frac{1}{a^2}(\bar{A}B)^2\ ,\\ &= \left[a^2 + \frac{(a,b)^2 - (a,\epsilon b)^2 + c^\mu c_\mu}{a^2}\right]\mathbb{1} \ + \ \frac{2(a,b)(a,\epsilon b)}{a^2}\,\gamma^{12} \ + \ \frac{2(a,b)c_\mu}{a^2}\,\gamma^\mu\\ &\equiv n_0\,\mathbb{1} + n_{12}\,\gamma^{12} + n_\mu\gamma^\mu\ , & (77)\\ N^{-1} &= \frac{1}{Z}(n_0\,\mathbb{1} - n_{12}\,\gamma^{12} - n_\mu\gamma^\mu)\ , \qquad Z \ = \ n_0^2 + n_{12}^2 - n_\mu n^\mu\ , & (78) \end{aligned}$$

for $M^{12}(p)\ \gamma^{12\,T}$ interchange A and B. Altogether I find (noting $a_\mu\epsilon^\mu_\nu n^\nu + n_{12}a_0 = 0$)

$$\begin{aligned} M^{\emptyset}(p) &= \frac{1}{Z}\Big[(n_0 a_\mu + a_0 n_\mu)\gamma^\mu \ + \ n_{12}a_\mu\epsilon^\mu_\nu\gamma^\nu \ - \ (n_\mu a^\mu + n_0 a_0)\mathbb{1}\Big]\ , & (79)\\ M^{12}(p) &= -\,M^{\emptyset}(p)\Big|_{a\leftrightarrow b}\ \gamma^{12}\ . & (80) \end{aligned}$$

In Table 1 the fermion matrix $m(y,y')$ following from Eqs. (60),(64),(79),(80) is evaluated for $0 \le y_\mu < 3, y' = 0$ and vice versa, putting $m_1 = 0.2$, $m_2 = 1$. The contributions of $M^{\emptyset}(y-y')$ are real, the contributions of $M^{12}(y-y')$ are purely imaginary. The vanishing of the M^{12}–part for couplings $y - y' = (n/2)e_\mu$, i.e. $\Pi^\mu(y-y') = R^{-2}_{\mu\nu}(y-y')$ follows from the behavior under geometric rotations $(R^{\mu\nu}_G)^2$ by 180°, which are symmetry transformations, and geometric reflections Π^μ_G, see Section 3. Under Π^μ_G the M^{12}-contributions pick up a minus sign, because this is the part of the action violating the d^1, d^2 flavor transformations and thus the geometric reflections.

It can be read off from Table 1 that the fermion matrix decomposes into a hermitian part m_+ coupling even sites with even sites, odd sites with odd sites, and an anti-hermitian part m_- coupling even sites with odd sites. The sign structure in Eq. (60) leads to

$$m_+(y, y-z)\ m_-(y-z, y') \ = \ m_-(y, y'+z)\ m_+(y'+z, y')\ , \qquad (81)$$

where it is crucial that $M^{\emptyset}(\bar{y} + e_{12}/2) = M^{12}(\bar{y}) = 0$. It follows $m_+m_- = m_-m_+$ and

$$m^\dagger m \ = \ m_+\, m_+ \ - \ m_-\, m_-\ , \qquad (82)$$

i.e. $m^\dagger m$ does not couple even and odd sites. This property is useful in numerical simulations with help of the pseudofermion method [17].

7 Relation to the cochain construction of staggered fermions

Here I will very shortly present the main features of Dirac-Kähler (DK) fermions [5] and their triangularization as lattice cochains. For a detailed description I refer to [4]. The DK equation

$$(d \; - \; \delta \; + \; m) \, \Phi \; = \; 0 \tag{83}$$

is equivalent to the Dirac equation for $N_f = 2^{d/2}$ degenerated flavors. $\Phi = \sum_H \varphi(x,H) dx^H$ is a inhomogeneous differential form, see Eq. (2), d is the external differentiation, δ the codifferentiation operator. The corresponding action is

$$S_{DK}[\bar{\Phi},\Phi] \; = \; [\bar{\Phi}, (d - \delta + m)\Phi] \; , \qquad \text{with } [\Phi,\Phi'] \equiv \sum_H \varphi(x,H) \, \varphi'(x,H) \; . \tag{84}$$

The unitary transformation in Eq. (4) of the component functions $\varphi(x,H)$ to the Dirac basis $\psi^b_y(x)$ leads back to the standard action Eq. (1).

A form may be considered as a mapping of all h–dimensional ($h = 1, \ldots, d$) areas $\mathcal{A}$ into $\mathbb{C}$

$$\Phi(\mathcal{A}) \; = \; \int_{\mathcal{A}} \Phi \; . \tag{85}$$

The lattice restriction of these forms arises by restriction of the integration areas $\mathcal{A}$ to lattice chains, i.e. combinations of h–dimensional lattice cells $[\bar{y},H]$ (sites, links, plaquettes, ...). The cell $[\bar{y},H]$ is spanned by the coarse lattice unit vectors $e_\mu, \mu \in H$ at the point $\bar{y} \in \bar{\Gamma}$. It is natural to represent this cell by the fine lattice point at its center $y = \bar{y} + e_H/2$. This decomposes the space of all forms into cochains, i.e. classes characterized by

$$\int_{[\bar{y},H]} \Phi \; = \; \phi(\bar{y} + e_H/2) \; . \tag{86}$$

Since the boundary of a lattice cell is again a lattice cell, the decomposition is consistent with the external differentiation d. From Stokes' theorem one obtains ($\Delta[\bar{y},H]$ is the oriented boundary of $[\bar{y},H]$)

$$\int_{[\bar{y},H]} d\Phi \; = \; \int_{\Delta[\bar{y},H]} \Phi \; \equiv \; (\tilde{\triangle}\phi) \, (\bar{y} + e_H/2) \; , \tag{87}$$

i.e. if Φ, Φ' are of the same class, so are $d\Phi, d\Phi'$. For the codifferentiation operator such a construction is not that clear, see [4]. However, its lattice correspondence $\tilde{\bigtriangledown}$ may be simply defined as the adjungated operator with respect to the lattice scalar product $(,)$, in order to preserve the usual anti-hermiticity property of $\tilde{\triangle} - \tilde{\bigtriangledown}$.

The transition from continuum forms to lattice cochains can be written by a blocking operator C mapping the component functions $\varphi(x,H)$ onto the blockspin variables $\phi(y) = \phi(\bar{y} + e_H/2)$

$$(C\varphi)(y) \; = \; \int_{[\bar{y},H]} \varphi(x,H) \; = \; \int_{[y]} \chi_y(x,H) \, \varphi(x,H) \; , \tag{88}$$

$$\chi_y(x,H) \; = \; \prod_{\mu \in H} (1/a) \; \prod_{\mu \notin H} \delta(x_\mu - y_\mu) \; . \tag{89}$$

This representation shows the similarity to the blocking scheme in Eq. (8), $[y]$ denotes the full coarse lattice hypercube with center y. The difference, however, is that the component functions $\varphi(x,H)$ are now averaged only over those directions μ with $\mu \in H$, in the other directions the

blocking scheme corresponds to decimation [20], as indicated by the characteristic functions $\chi_y(x, H)$. Now Eq. (87) reads $C\,d = \tilde{\triangle}\,C$, and the lattice restriction of the DK action Eq. (84) might be written

$$\begin{aligned} & S_{DK}[\bar{\Phi}, \Phi] &=& [\bar{\Phi}, d\Phi] - [d\bar{\Phi}, \Phi] + m[\bar{\Phi}, \Phi] & (90) \\ \longrightarrow \quad & S[C\bar{\Phi}, C\Phi] &=& (C\bar{\Phi}, Cd\Phi) - (Cd\bar{\Phi}, C\Phi) + m(C\bar{\Phi}, C\Phi) & \\ & &=& (C\bar{\Phi}, (\tilde{\triangle} - \tilde{\bigtriangledown} + m)C\Phi) & (91) \end{aligned}$$

As described in [4], the result $S[\bar{\phi}, \phi]$ is the staggered fermion action of Eq. (45).

One may ask whether the cochain blocking given by C in Eq. (88) is a reasonable alternative blockspin definition in the sense of blocking from the continuum. Unfortunately, as prescription in this scheme, cochain blocking violates the discrete flavor transformation symmetry of staggered fermions, which has proven quite useful for the calculation of the perfect action. Consider for instance the propagator of two blockspins with the same spin-flavor content

$$u_H(\bar{y}) \;=\; \langle \phi(\bar{y} + e_H/2)\, \bar{\phi}(e_H/2) \rangle \;. \tag{92}$$

For its evaluation along the lines of Eq. (4) one has to integrate the continuum fields over different lattice cells of dimension h depending on H. This destroys the discrete modified flavor symmetry, which in this case requires H–independence of $u_H(\bar{y})$. So, in this naive way, a direct exploitation of the cochain blocking scheme seems not to be useful for RG considerations. Nevertheless it seems me worthwhile to look for a more clever combination of these two approaches to a lattice fermion formulation.

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